JOURNAL OF APPROXIMATION THEORY 2, 355–373 (1969)

On Simultaneous Approximation and Interpolation which Preserves the Norm[1]

FRANK DEUTSCH AND PETER D. MORRIS

Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802

1. INTRODUCTION

In [8], H. Yamabe established the following "simultaneous approximation and interpolation" theorem, which generalized a result of Walsh ([6]; p. 310) (cf. also [1]), and is related to a theorem of Helly in the theory of moments (cf. [2]; pp. 86–87):

THEOREM (Yamabe). *Let M be a dense convex subset of the real normed linear space X, and let $x_1^*, \ldots, x_n^* \in X^*$. Then for each $x \in X$, and each $\epsilon > 0$, there exists a $y \in M$ such that $\|x - y\| < \epsilon$ and $x_i^*(y) = x_i^*(x)$ $(i = 1, \ldots, n)$.*

Wolibner [7], in essence, proved that Yamabe's theorem could be sharpened in the particular case where $X = C([a,b])$, $M = \mathscr{P} =$ the set of polynomials, and where the x_i^* are "point evaluations". Indeed, from the results of [7], one can readily deduce the following

THEOREM (Wolibner). *Let $a \leqslant t_1 < t_2 < \ldots < t_n \leqslant b$, and let $\mathscr{P}$ be the set of polynomials. Then for each $x \in C([a,b])$, and each $\epsilon > 0$, there exists a $p \in \mathscr{P}$ such that $\|x - p\| < \epsilon$, $p(t_i) = x(t_i)$ $(i = 1, \ldots, n)$, and $\|p\| = \|x\|$.*

Motivated by Wolibner's theorem, we consider the following more general problem. Let M be a dense subspace of the normed linear space X, and let $\{x_1^*, \ldots, x_n^*\}$ be a finite subset of the dual space X^*. The triple $(X, M, \{x_1^*, \ldots, x_n^*\})$ will be said to have property SAIN (simultaneous approximation and interpolation which is norm-preserving) provided that the following condition is satisfied:

For each $x \in X$, and each $\epsilon > 0$, there exists a $y \in M$ such that

$$\|x - y\| < \epsilon, \qquad x_i^*(y) = x_i^*(x) \qquad (i = 1, \ldots, n), \quad \text{and} \quad \|y\| = \|x\|.$$

[1] Supported by grants from the National Science Foundation.

It is not hard to give examples of triples $(X, M, \{x_1^*, \ldots, x_n^*\})$ which do not have property SAIN. On the other hand, for those triples which do have property SAIN, we have a stronger conclusion that can be gleaned from Yamabe's theorem. It is the purpose of this paper to determine necessary and sufficient conditions in order that a triple have property SAIN. After establishing some useful results of a general nature in §2, we prove (Theorem 3.2) that if M is a dense subspace of a Hilbert space X, and if $x_1^*, \ldots, x_n^* \in X^*$, then $(X, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN if and only if each x_i^* attains its norm on the unit ball of M. In §4 we consider the case $X = C(T)$, T compact Hausdorff. The main result here (Theorem 4.1) is that $(C(T), M, \{x_1^*, .., x_n^*\})$ has property SAIN if M is a dense subalgebra (or dense linear sublattice containing constants), and the x_i^* are point evaluations. This result contains that of Wolibner, and represents a strengthening of the Stone–Weierstrass theorem (Corollary 4.2). Various counterexamples are constructed, which show that these results cannot be extended very far. In §5, we consider a particular dense subspace of $X = L_p$ for $1 < p < \infty$, and in §6, a particular dense subspace of $X = L_1$. We also pose a few open problems.

It is worth mentioning here some specific results which are somewhat related to our problem. We first remark that Wolibner actually showed that if $x(t_{i+1}) \neq x(t_i)$, then the polynomial p of the conclusion of his theorem can be chosen to be monotone in each of the intervals $[t_i, t_{i+1}]$. S. Young [9], independently, gave an elegant proof of this latter fact. Paszkowski ([4]; p. 8) dropped the approximation part of the conclusion of Wolibner's theorem (i.e., he sought only a polynomial p such that $p(t_i) = x(t_i)$ $(i = 1, \ldots, n)$, and $\|p\| = \|x\|$), and showed that in this case the *degree* of the polynomial p which works is *independent* of the function x being interpolated, and depends only on the points t_i! Singer [5] extended Yamabe's theorem to the case where X is a real linear topological space. Also, in [1] there was given a different proof of Yamabe's theorem, where M is a subspace in any (real or complex) linear topological space. We now observe that the following theorem, which encompasses the main results of both [5] and [1], is true.

THEOREM 1.1. *Let M be a dense convex subset of the (real or complex) linear topological space X, and let $f_1, \ldots, f_n$ be continuous linear functionals on X. Then for each $x \in X$, and each neighborhood U of x, there exists a $y \in M$ such that $y \in U$, and $f_i(y) = f_i(x)$ $(i = 1, \ldots, n)$.*

The proof of Theorem 1.1 is exactly the same as in [5]. We recall that the essential fact used there was the well-known result that if M is a dense convex subset of the real linear topological space X, and Y is a subspace of X having finite codimension, then $M \cap Y$ is dense in Y. We need only observe that this fact is also true in case X is a complex linear topological space. To see this, we employ the standard trick of regarding X as a real space, and then note that if Y has (finite) codimension n in the complex space X, then Y has (finite) codimension $2n$ in the real space X.

We conclude the introduction by reviewing some notation and terminology. Throughout the paper X will denote a real normed linear space (although some of our results are valid in complex spaces, as well), X^* will denote the dual of X, i.e., the Banach space of all continuous linear functionals x^* on X, with the norm $\|x^*\| = \sup\{|x^*(x)| : \|x\| \leqslant 1\}$. The closed unit ball of a normed linear space Y, denoted $S(Y)$, is the set $\{y \in Y : \|y\| \leqslant 1\}$. A functional $x^* \in X^*$ is said to *attain its norm* on $S(X)$, if there is an element $x \in S(X)$ such that $x^*(x) = \|x^*\|$. By *subspace* we shall always mean linear subspace. All other notation and terminology will conform to those in [2].

2. General Results

LEMMA 2.1. *Let M be a dense subspace of X, and let Γ be a finite-dimensional subspace of X^*. Let $x \in X$, $\epsilon > 0$, and suppose there exists a $y_1 \in M$ such that $\|x - y_1\| < \epsilon$, $x^*(y_1) = x^*(x)$ for all $x^* \in \Gamma$, and $\|y_1\| < \|x\|$. Then there exists a $y_2 \in M$ such that $\|x - y_2\| < \epsilon$, $x^*(y_2) = x^*(x)$ for all $x^* \in \Gamma$, and $\|y_2\| > \|x\|$.*

Proof. Let $\|x - y_1\| = \lambda\epsilon$ where $0 < \lambda < 1$, and set $z = 2x - y_1$. Then $\|x - z\| = \|y_1 - x\| = \lambda\epsilon$, $x^*(z) = 2x^*(x) - x^*(y_1) = x^*(x)$ for all $x^* \in \Gamma$, and $\|z\| = \|2x - y_1\| \geqslant 2\|x\| - \|y_1\| > \|x\|$. By Yamabe's theorem, we can choose $y_2 \in M$ such that $x^*(y_2) = x^*(z) (= x^*(x))$ for all $x^* \in \Gamma$, and $\|z - y_2\| < \min\{(1 - \lambda)\epsilon, \|z\| - \|x\|\}$. Then

$$\|x - y_2\| \leqslant \|x - z\| + \|z - y_2\| < \lambda\epsilon + (1 - \lambda)\epsilon = \epsilon,$$

and

$$\|y_2\| = \|y_2 - z + z\| \geqslant \|z\| - \|z - y_2\| > \|z\| - (\|z\| - \|x\|) = \|x\|,$$

which completes the proof.

LEMMA 2.2. *Let X, M, and Γ be as in Lemma 2.1. Let $x \in X$, $\epsilon > 0$, and suppose there exist y_1, y_2 in M such that $\|x - y_1\| < \epsilon$, $x^*(y_1) = x^*(x)$ for all $x^* \in \Gamma$, and $\|y_1\| < \|x\| < \|y_2\|$. Then there exists a $y \in M$ such that $\|x - y\| < \epsilon$, $x^*(y) = x^*(x)$ for all $x^* \in \Gamma$, and $\|y\| = \|x\|$.*

Proof. For each $\lambda \in [0, 1]$, define $y_\lambda = \lambda y_2 + (1 - \lambda) y_1$. Then $y_\lambda \in M$ for each $\lambda \in [0, 1]$, and the function $f(\lambda) = \|y_\lambda\|$ is continuous on $[0, 1]$. Since $f(0) = \|y_1\|$, and $f(1) = \|y_2\|$, it follows that there is a $\lambda_0 \in (0, 1)$ such that $f(\lambda_0) = \|x\|$, i.e., $\|y_{\lambda_0}\| = \|x\|$. Also,

$$\begin{aligned}\|x - y_{\lambda_0}\| &= \|\lambda_0(x - y_2) + (1 - \lambda_0)(x - y_1)\| \\ &\leqslant \lambda_0\|x - y_2\| + (1 - \lambda_0)\|x - y_1\| < \epsilon.\end{aligned}$$

Finally, for each $x^* \in \Gamma$,

$$x^*(y_{\lambda_0}) = \lambda_0 x^*(y_2) + (1-\lambda_0) x^*(y_1) = \lambda_0 x^*(x) + (1-\lambda_0) x^*(x)$$
$$= x^*(x).$$

Taking $y = y_{\lambda_0}$ completes the proof.

As an immediate consequence of Lemmas 2.1 and 2.2, we obtain

LEMMA 2.3. *Let M be a dense subspace of X, and let Γ be a finite-dimensional subspace of X^*. Let $x \in X$, $\epsilon > 0$, and suppose there is a $y_1 \in M$ such that $\|x - y_1\| < \epsilon$, $x^*(y_1) = x^*(x)$ for all $x^* \in \Gamma$, and $\|y_1\| \leqslant \|x\|$. Then there exists $y \in M$ such that $\|x - y\| < \epsilon$, $x^*(y) = x^*(x)$ for all $x^* \in \Gamma$, and $\|y\| = \|x\|$.*

Remark 2.1 Lemma 2.3 can be reworded as follows: $(X, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN if and only if for each $x \in X$, and each $\epsilon > 0$, there exists $y \in M$ such that $\|x - y\| < \epsilon$, $x_i^*(y) = x_i^*(x)$ for $i = 1, \ldots, n$, and $\|y\| \leqslant \|x\|$.

The following is a helpful tool which will be used throughout the sequel.

LEMMA 2.4. *Let M be a dense subspace of X, and let $x^* \in X^*$, $\|x^*\| = 1$. Let $x \in X$, $\|x\| = 1$, and suppose that $|x^*(x)| < 1$. Then for each $\epsilon > 0$, there exists $y \in M$ such that $\|x - y\| < \epsilon$, $x^*(y) = x^*(x)$, and $\|y\| = \|x\|$.*

Proof. It suffices to consider the case $0 \leqslant x^*(x) < 1$. Choose $x_0 \in X$ such that $\|x_0\| < 1$, and $x^*(x) < x^*(x_0)$. Select any $\lambda \in (0,1)$ with $1 - \epsilon\|x - x_0\|^{-1} < \lambda$, and set $x^+ = \lambda x + (1-\lambda)x_0$. Then $\|x^+\| \leqslant \lambda\|x\| + (1-\lambda)\|x_0\| < 1$,

$$x^*(x^+) = \lambda x^*(x) + (1-\lambda)\, x^*(x_0) > x^*(x),$$

and

$$\|x - x^+\| = \|(1-\lambda)(x - x_0)\| = (1-\lambda)\|x - x_0\| < \epsilon.$$

Thus, letting $S(z; r)$ denote the open sphere centered at z, with radius r (i.e., the set $\{w \in X : \|w - z\| < r\}$), and $H^+ = \{z \in X : x^*(z) > x^*(x)\}$, we have shown that the open set

$$U^+ = S(0;1) \cap S(x;\epsilon) \cap H^+$$

is not empty. Similarly, the open set

$$U^- = S(0;1) \cap S(x;\epsilon) \cap H^-$$

is not empty, where $H^- = \{z \in X : x^*(z) < x^*(x)\}$. (In fact, the element $x^- = \alpha x + (1-\alpha)(-x_0)$, where $\alpha \in (0,1)$, and $1 - \epsilon\|x + x_0\|^{-1} < \alpha$, is in U^-.) Since M is dense, we can choose points $y^+ \in M \cap U^+$, and $y^- \in M \cap U^-$. Let

$\gamma \in (0,1)$ be chosen so that the element $y = \gamma y^+ + (1-\gamma)y^-$ satisfies $x^*(y) = x^*(x)$. Clearly, $\|y\| < 1$, and

$$\begin{aligned}\|x-y\| &= \|\gamma(x-y^+) + (1-\gamma)(x-y^-)\| \\ &\leqslant \gamma\|x-y^+\| + (1-\gamma)\|x-y^-\| < \epsilon.\end{aligned}$$

An appeal to Lemma 2.3 completes the proof.

Remark 2.2. The hypothesis $|x^*(x)| < 1$, in Lemma 2.4, is essential. Indeed, the following example shows that the conclusion of Lemma 2.4 does not hold, in general, if $|x^*(x)| = 1$. Let $X = C([0,1])$, and let

$$M = \{x \in C([0,1]) : x'(\tfrac{1}{2}) \text{ exists, } x'(\tfrac{1}{2}) = x(0) - x(1)\}.$$

It is not hard to see that M is a dense subspace of $C([0,1])$. Define x^* on X by $x^*(x) \equiv x(\frac{1}{2})$. Then $x^* \in X^*$, and $\|x^*\| = 1$. Let $x_0 \in X$ be defined by $x_0(t) = 1$ if $0 \leqslant t \leqslant \frac{1}{2}$, and $x_0(t) = -2t + 2$ if $\frac{1}{2} < t \leqslant 1$. Then $\|x_0\| = 1$, and $x^*(x_0) = 1$. Clearly, if $y \in M$, $x^*(y) = x^*(x_0)$, and $\|y\| = \|x_0\|$, then $y'(\frac{1}{2}) = 0$. In particular, $y(0) = y(1)$. It follows that

$$\max\{|x_0(0) - y(0)|, |x_0(1) - y(1)|\} \geqslant \tfrac{1}{2},$$

and hence $\|x_0 - y\| \geqslant \frac{1}{2}$. Thus, for $x = x_0$, and $0 < \epsilon \leqslant \frac{1}{2}$, it is not possible to find a $y \in M$ satisfying the conclusion of Lemma 2.4.

Remark 2.3. Unfortunately, Lemma 2.4 cannot be extended so as to be valid if there were more than one norm-one functional x^* satisfying $|x^*(x)| < 1$. In fact, we can even prove somewhat more. Namely,

Proposition 2.1. *There is a dense subspace M of l_1, functionals x^*, y^* in $l_1{}^*$ with $\|x^*\| = \|y^*\| = 1$, and an element $x \in l_1$ with $\|x\| = 1$, $|x^*(x)| < 1$, and $|y^*(x)| < 1$, having the property that it is* not *possible to find a $y \in M$ such that $\|y\| = \|x\|$, $x^*(y) = x^*(x)$, and $y^*(y) = y^*(x)$.*

Proof. As usual, we identify $l_1{}^*$ with the sequence space l_∞. Let

$$M = \left\{y = (\eta_1, \eta_2, \ldots) \in l_1 : \sum_1^\infty n\eta_n = 0\right\}, \qquad x^* = (1, -1, 1, 0, 0, \ldots) \in l_\infty,$$

$$y^* = (-1, 1, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots) \in l_\infty, \qquad \text{and} \qquad x = (0, 0, 0, \tfrac{1}{2}, (\tfrac{1}{2})^2, (\tfrac{1}{2})^3, \ldots).$$

M is clearly a subspace of l_1. To see that M is dense in l_1, it suffices to show that each of the unit vectors

$$e_k = (\delta_{1k}, \delta_{2k}, \ldots) \qquad (k = 1, 2, \ldots)$$

can be approximated arbitrarily well by elements of M (since span $\{e_1, e_2, \ldots\}$ is dense in l_1). Fix an index k, and let $\epsilon > 0$. Choose $n > k$ so that $n^{-1} < \epsilon$. Then the vector

$$y = (\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{nk-k-1}, -n^{-1}, 0, \ldots)$$

is in M, and $\|e_k - y\| = n^{-1} < \epsilon$. Thus, M is dense. Now, $\|x^*\| = \|y^*\| = 1$, $x \in l_1$, and $\|x\| = \sum_1^\infty (\frac{1}{2})^k = 1$. Also, $x^*(x) = 0$, and $y^*(x) = \sum_1^\infty (\frac{1}{2})^{k+1} = \frac{1}{2}$. Let $y = (\eta_1, \eta_2, \ldots) \in M$, and suppose that $x^*(y) = x^*(x)$, $y^*(y) = y^*(x)$, and $\|y\| = \|x\| = 1$. It follows that

$$\eta_1 - \eta_2 + \eta_3 = 0, \qquad -\eta_1 + \eta_2 - \tfrac{1}{2}\eta_3 + \tfrac{1}{2}\sum_4^\infty \eta_n = \tfrac{1}{2},$$

and $\sum_1^\infty |\eta_n| = 1$. Solving for η_2 from the first equation, and substituting into the second, we deduce that $\sum_3^\infty \eta_n = 1$. This, along with the third equation, implies that $\eta_1 = \eta_2 = \eta_3 = 0$, and $\eta_n \geqslant 0$ for all n. But $y \in M$, so that $\sum_1^\infty n\eta_n = 0$, and, hence, $\eta_n = 0$ for all n, i.e., $y = 0$, which is a contradiction. This completes the proof.

By examining the steps of the proof, we observe that the proposition is valid for any dense subspace M of l_1 which does not contain any *positive* elements, i.e., nonzero elements $y = (\eta_1, \eta_2, \ldots)$ such that $\eta_n \geqslant 0$ for all n.

The following two theorems follow rather easily from Lemma 2.4.

THEOREM 2.1. *Let M be a dense subspace of $X, x^* \in X^*$, and suppose that either x^* does not attain its norm on $S(X)$, or x^* attains its norm on $S(X)$ only at points in M. Then $(X, M, \{x^*\})$ has property SAIN.*

Proof. Let $x \in X$, and $\epsilon > 0$. We can assume that $\|x^*\| = \|x\| = 1$, and $x \notin M$. By hypothesis, $|x^*(x)| < 1$. By Lemma 2.4, there exists a $y \in M$ such that $\|x - y\| < \epsilon$, $x^*(y) = x^*(x)$, and $\|y\| = \|x\|$. This completes the proof.

Remark 2.4. The converse of Theorem 2.1 is false. For example, as a consequence of Theorem 4.1 below, it follows that there are triples $(X, M, \{x^*\})$ with property SAIN, and such that x^* attains its norm at points in $S(M)$, as well as at points in $S(X) \sim M$.

THEOREM 2.2. *Let M be a dense subspace of the strictly convex normed linear space X, let $x^* \in X^*$, and suppose that x^* attains its norm on $S(M)$. Then $(X, M, \{x^*\})$ has property SAIN.*

Proof. By strict convexity, x^* must attain its norm at a unique point of M, so that Theorem 2.1 applies.

Remark 2.5. The hypothesis that X be strictly convex, in Theorem 2.2, cannot be dropped. In fact, the same example as in Remark 2.2 establishes this fact.

THEOREM 2.3. *Let M be a dense subspace of the normed linear space X, and let $x_1^*,\ldots,x_n^* \in X^*$. A necessary condition that $(X, M, \{x_1^*,\ldots,x_n^*\})$ have property SAIN is that each x_i^* either attains its norm on $S(M)$, or does not attain its norm on $S(X)$ at all.*

Proof. If some x_i^* attained its norm at a point $x \in S(X) \sim M$, but not on $S(M)$, then we would have, in particular, that $x_i^*(y) < \|x_i^*\| = x_i^*(x)$ for all $y \in S(M)$. This contradicts property SAIN, and completes the proof.

In the case of a strictly convex space, and one interpolation condition, i.e., $n = 1$, the necessary condition of Theorem 2.3 is also sufficient, as a consequence of Theorem 2.2. Thus we have the following.

COROLLARY 2.1. *Let M be a dense subspace of the strictly convex normed linear space X, and let $x^* \in X^*$. Then $(X, M, \{x^*\})$ has property SAIN if and only if either x^* attains its norm on $S(M)$, or x^* does not attain its norm on $S(X)$ at all.*

In a reflexive Banach space X, it is well known that every $x^* \in X^*$ attains its norm on $S(X)$. (Indeed, this property characterizes reflexive Banach spaces [3]; Theorem 5].) Thus we immediately obtain the following corollary of Theorem 2.3, which we state for future reference.

COROLLARY 2.2. *Let M be a dense subspace of the reflexive Banach space X, and let $x_1^*,\ldots,x_n^* \in X^*$. A necessary condition that $(X, M, \{x_1^*,\ldots,x_n^*\})$ have property SAIN is that each x_i^* attains its norm on $S(M)$.*

Combining Corollaries 2.1 and 2.2, we deduce

COROLLARY 2.3. *Let M be a dense subspace of the strictly convex reflexive Banach space X, and let $x^* \in X^*$. Then $(X, M, \{x^*\})$ has property SAIN if and only if x^* attains its norm on $S(M)$.*

Remark 2.6. It is an interesting open question whether the necessary condition of Corollary 2.2 is also sufficient in the case $n > 1$. (As we shall see in §3, the answer is in the affirmative if X is a Hilbert space. Also, we shall see in §5 that the answer is affirmative for a certain subspace M of L_p $(1 < p < \infty)$.)

3. The Hilbert Space Case

In the case where X is a Hilbert space, we can give a complete characterization of those triples having property SAIN (Theorem 3.2). It can be easily deduced from

THEOREM 3.1. *Let M be a dense subspace of the Hilbert space X, and let $y_1,\ldots,y_n \in M$. Then for each $x \in X$, and each $\epsilon > 0$, there exists $y \in M$ such that* $\|x-y\| < \epsilon$, $\langle y,y_i\rangle = \langle x,y_i\rangle$ $(i=1,\ldots,n)$, and $\|y\| = \|x\|$.

Proof. It is no loss of generality to assume that the y_i are linearly independent. By replacing the y_i by an orthonormal basis for $Y \equiv \text{span}\{y_1,\ldots,y_n\}$, we can, in fact, assume that the y_i are orthonormal. Let $x \in X$, and $\epsilon > 0$. Since $X = Y \oplus Y^{\perp}$ ([2]; p. 249), we can write $x = \sum_1^n \alpha_i y_i + z$, where $\alpha_i = \langle x,y_i\rangle$ and $z \in Y^{\perp}$. Since $Y^{\perp}$ has finite codimension, $M \cap Y^{\perp}$ is dense in $Y^{\perp}$ (cf. the remark following Theorem 1.1). Thus we can choose $w \in M \cap Y^{\perp}$ so that $\|z-w\| < \epsilon$, and $\|w\| \leqslant \|z\|$. Then, setting $y = \sum_1^n \alpha^i y_i + w$, we have that $y \in M$, $\|y-x\| = \|z-w\| < \epsilon$, $\langle y,y_i\rangle = \alpha_i = \langle x,y_i\rangle$ $(i=1,\ldots,n)$, and $\|y\|^2 = \|\sum_1^n \alpha_i y_i\|^2 + \|w\|^2 \leqslant \|\sum_1^n \alpha_i y_i\|^2 + \|z\|^2 = \|x\|^2$. An appeal to Lemma 2.3 completes the proof.

THEOREM 3.2. *Let M be a dense subspace of the Hilbert space X, and let $x_1^*,\ldots,x_n^* \in X^*$. Then $(X,M,\{x_1^*,\ldots,x_n^*\})$ has property SAIN if and only if each x_i^* attains its norm on $S(M)$.*

Proof. The necessity is a consequence of Corollary 2.2. To prove the sufficiency, we first observe that corresponding to each x_i^*, there is a unique $y_i \in X$ so that $x_i^*(x) = \langle x,y_i\rangle$ for all $x \in X$, and $\|y_i\| = \|x_i^*\|$ (cf., e.g., [2]; p. 249). Letting $m_i \in S(M)$ denote the point where x_i^* attains its norm, we have that $\|x_i^*\| = x_i^*(m_i)$ $(i=1,\ldots,n)$, i.e. $\|y_i\| = \langle m_i,y_i\rangle$ $(i=1,\ldots,n)$. By the condition for equality in Schwarz's inequality ([2]; p. 248), we deduce that $y_i = \|y_i\| m_i \in M$ $(i=1,\ldots,n)$. An application of Theorem 3.1 now completes the proof.

4. The Case of Spaces of Type C(T)

Throughout this section, T will denote a compact Hausdorff space, and $C(T)$ the real-valued continuous functions on T, endowed with the supremum norm. If $t \in T$, we define δ_t to be the linear functional "evaluation at t", i.e., $\delta_t(x) = x(t)$ for all $x \in C(T)$.

LEMMA 4.1. *Let M be a dense subspace of $C(T)$, and let $t_1,\ldots,t_n \in T$. Let $x \in C(T)$ satisfy $|x(t_i)| < \|x\|$ $(i=1,\ldots,n)$. Then for each $\epsilon > 0$ there exists a $y \in M$ such that $\|x-y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i=1,\ldots,n)$, and $\|y\| = \|x\|$.*

Proof. We may assume that $\|x\| = 1$. By Lemma 2.3, it suffices to show that there is a $y \in M$ such that $\|x - y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1,\ldots,n)$, and $\|y\| \leqslant 1$. We proceed by induction on n. For $n = 1$, the result is a consequence of Lemma 2.4. Now suppose the conclusion holds for N or fewer points t_i, and let $t_1,\ldots,t_{N+1} \in T$. By hypothesis, there exists a $y_1 \in M$ such that $\|x - y_1\| < \epsilon$, $y_1(t_i) = x(t_i)$ $(i = 1, \ldots, N)$, and $\|y_1\| \leqslant 1$. Let $c = x(t_{N+1})$. If $y_1(t_{N+1}) = c$, we are done. Suppose, then, that $y_1(t_{N+1}) \neq c$. We can assume that $y_1(t_{N+1}) > c$, since the other case is similar. Let $\epsilon_1 = \min\{\epsilon, 1 - |c|\}$. Let V be an (open)neighborhood of t_{N+1} such that $t_i \notin V$ for $i = 1,\ldots,N$, and $|x(t) - c| < \epsilon_1/2$ if $t \in V$. By Urysohn's lemma, we can choose $h \in C(T)$ such that $h(t_{N+1}) = -\epsilon_1/2$, $h(t) = 0$ if $t \notin V$, and $-\epsilon_1/2 \leqslant h(t) \leqslant 0$ for all $t \in T$. Then $\|x + h\| = 1$, $(x + h)(t_i) = x(t_i)$ $(i = 1,\ldots,N)$, and $(x + h)(t_{N+1}) = c - \epsilon_1/2$. Choose $y_2 \in M$ such that $\|y_2 - (x + h)\| < \epsilon_1/2$, $y_2(t_i) = (x + h)(t_i) = x(t_i)$ $(i = 1, \ldots, N)$, and $\|y_2\| \leqslant \|x + h\| = 1$. Then $y_2(t_{N+1}) \leqslant c$, and

$$\|y_2 - x\| \leqslant \|y_2 - (x + h)\| + \|h\| < \epsilon_1/2 + \epsilon_1/2 = \epsilon_1 \leqslant \epsilon.$$

Let y be the convex combination of y_1 and y_2 such that $y(t_{N+1}) = c$. Then $y \in M$, $\|x - y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1,\ldots,N + 1)$, and $\|y\| \leqslant 1$. This completes the induction and the proof.

Remark 4.1. From Lemma 4.1, it is tempting to conclude that $(C(T), M, \{\delta_{t_1},\ldots,\delta_{t_n}\})$ has property SAIN. However, we recall that in Remark 2.2 we gave an example of a dense subspace M of $C([0,1])$ which contained constants, but such that $(C([0,1]), M, \delta_{1/2})$ did *not* have property SAIN.

On the other hand, if M is a dense *subalgebra of* $C(T)$, then $(C(T), M, \{\delta_{t_1},\ldots,\delta_{t_n}\})$ *does* have property SAIN.

THEOREM 4.1. *Let A be a dense subalgebra of $C(T)$, and let $t_1,\ldots,t_n \in T$. Then $(C(T), A, \{\delta_{t_1},\ldots,\delta_{t_n}\})$ has property SAIN.*

Proof. We need the following

LEMMA 4.2. *Let A and $\{t_i\}$ be as in the theorem. For each $\epsilon > 0$, there exists an element $e \in A$ such that $\|e - 1\| < \epsilon$, $e(t_i) = 1$ $(i = 1,\ldots,n)$, and $e \leqslant 1$.*

Proof of Lemma. By Yamabe's theorem we can choose a $y \in A$ such that $\|y - 1\| < \sqrt{\epsilon}$, and $y(t_i) = 1$ $(i = 1,\ldots,n)$. Then $\|y^2 - 2y + 1\| < \epsilon$. Let $e = 2y - y^2$. Then $e \in A$, $\|e - 1\| < \epsilon$, and $e(t_i) = 1$ $(i = 1,\ldots,n)$. Also, $y^2 - 2y + 1 \geqslant 0$ implies that $e = 2y - y^2 \leqslant 1$. This proves the lemma.

To prove the theorem, let $x \in C(T)$, and $\epsilon > 0$. We can assume that $\|x\| = 1$ and $\epsilon < 1$. By Lemma 2.3, it suffices to show the existence of a $y \in A$ such that $\|x - y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1,\ldots,n)$, and $\|y\| \leqslant 1$. We proceed by induction on n. For $n = 1$, let $c = x(t_1)$.

Case 1. $|c| < 1$.

This is just a special case of Lemma 4.1.

Case 2. $|c| = 1$.

We may assume that $c = 1$. Choose $\epsilon_1 > 0$ so that $2\sqrt{2}\epsilon_1 + \epsilon_1{}^2 < \epsilon/4$. Let $z(t) = \sqrt{1 - x(t)}$. By Yamabe's theorem, we can choose $y_1 \in A$ so that $\| y_1 - z \| < \epsilon_1$, and $y_1(t_1) = z(t_1) = 0$. Then

$$\| y_1{}^2 - z^2 \| \leqslant (\| y_1 \| + \| z \|) \| y_1 - z \| \leqslant (\sqrt{2} + \epsilon_1 + \sqrt{2})\,\epsilon_1 \qquad (4.1)$$
$$= 2\sqrt{2}\epsilon_1 + \epsilon_1{}^2 < \epsilon/4.$$

Let $\lambda = 2(2 + \epsilon/2)^{-1}$. Then

$$\| \lambda y_1{}^2 - z^2 \| \leqslant \lambda \| y_1{}^2 - z^2 \| + (1 - \lambda) \| z^2 \|$$
$$< \epsilon/4 + 2\epsilon/4 = 3\epsilon/4.$$

Choose $\epsilon_2 > 0$ so that $\epsilon_2 < 2 - 2(2 + \epsilon/4)(2 + \epsilon/2)^{-1}$. By Lemma 4.1, there exists $e \in A$ such that $\| e - 1 \| < \min\{\epsilon/4, \epsilon_2\}$, $e(t_1) = 1$, and $e \leqslant 1$. Set $y = e - \lambda y_1{}^2$. Then $y \in A$, and $y(t_1) = 1 = x(t_1)$. Also,

$$\| y - x \| = \| e - \lambda y_1{}^2 - 1 + z^2 \| \leqslant \| e - 1 \| + \| z^2 - \lambda y_1{}^2 \| < \epsilon/4 + 3\epsilon/4 = \epsilon.$$

Moreover, $\| x \| = 1$ implies $0 \leqslant 1 - x \leqslant 2$ (i.e., $0 \leqslant z^2 \leqslant 2$), and so $0 \leqslant y_1{}^2 \leqslant 2 + \epsilon/4$, from eq. (4.1). Then

$$0 \leqslant \lambda y_1{}^2 \leqslant 2(2 + \epsilon/2)^{-1}(2 + \epsilon/4) < 2 - \epsilon_2.$$

Finally,

$$-1 < -1 + (e - 1 + \epsilon_2) \leqslant e - \lambda y_1{}^2 \leqslant e \leqslant 1.$$

In particular, $\| y \| \leqslant 1$. This proves the theorem, in case $n = 1$.

Now suppose the conclusion holds for N or fewer points t_i, and let $t_1, \ldots, t_{N+1} \in T$. Let $c = x(t_{N+1})$. By hypothesis, there exists a $y_1 \in A$ such that $\| x - y_1 \| < \epsilon$, $y_1(t_i) = x(t_i)$ $(i = 1, \ldots, N)$, and $\| y_1 \| \leqslant 1$. If $y_1(t_{N+1}) = c$, we are done. Suppose, then, that $y_1(t_{N+1}) \neq c$. We assume that $y_1(t_{N+1}) > c$, since the other case is similar. Since $\| y_1 \| \leqslant 1$, $-1 \leqslant c < 1$.

Case 1. $-1 < c < 1$.

Proceeding exactly as in the proof of Lemma 4.1, we deduce the existence of a $y \in A$ such that $\| x - y \| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1, \ldots, N + 1)$, and $\| y \| \leqslant 1$.

Case 2. $c = -1$.

Let $z(t) = \sqrt{1 + x(t)}$. Then $z(t_{N+1}) = 0$. By case 1, we can find a $y_1 \in A$ such that $\| y_1 - z \| < \epsilon/4\sqrt{2}$, $y_1(t_i) = z(t_i)$ $(i = 1, \ldots, N + 1)$, and $\| y_1 \| \leqslant \| z \| \leqslant \sqrt{2}$. In particular,

$$\| y_1{}^2 - (1 + x) \| = \| y_1{}^2 - z^2 \| \leqslant (\| y_1 \| + \| z \|) \| y_1 - z \|$$
$$< 2\sqrt{2}\,\epsilon/4\sqrt{2} = \epsilon/2,$$

and $0 \leqslant y_1^2 \leqslant 2$. By Lemma 4.1, we can choose an $e \in A$ so that $\|e - 1\| < \epsilon/8$, $e(t_i) = 1$ $(i = 1,\ldots,N+1)$, and $e \leqslant 1$. Let $y = ey_1^2 - e$. Then $y \in A$,

$$\begin{aligned}\|x - y\| &= \|x + 1 - y_1^2 + (y_1^2 - 1)(1 - e)\| \\ &\leqslant \|x + 1 - y_1^2\| + (\|y_1^2\| + 1)\|1 - e\| \\ &< \epsilon/2 + 3\epsilon/8 < \epsilon,\end{aligned}$$

and $y(t_i) = x(t_i)$ $(i = 1,\ldots,N+1)$. Now $0 \leqslant y_1^2 \leqslant 2$, so that $-1 \leqslant y_1^2 - 1 \leqslant 1$. Since $\|e - 1\| < \epsilon/8$, and $\epsilon < 1$, we have $e(t) > 0$ for all $t \in T$. Thus,

$$-1 \leqslant -e \leqslant e(y_1^2 - 1) \leqslant e \leqslant 1,$$

and so $\|y\| = \|e(y_1^2 - 1)\| \leqslant 1$.

This completes the induction and hence the proof.

We can strengthen the conclusion of Theorem 4.1 for those functions in $C(T)$ which are nonnegative.

COROLLARY 4.1. *Let A be a dense subalgebra of $C(T)$, and let $t_1,\ldots,t_n \in T$. Then for each nonnegative function $x \in C(T)$, and each $\epsilon > 0$, there exists a nonnegative $y \in A$ such that $\|x - y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1,\ldots,n)$, and $\|y\| = \|x\|$.*

Proof. Let $x \in C(T)$, $x(t) \geqslant 0$ for all $t \in T$, and $\epsilon > 0$. We can assume $\|x\| = 1$. Applying Theorem 4.1 to the function $\sqrt{x}$, we obtain a $y_1 \in A$ such that $\|\sqrt{x} - y_1\| < \epsilon/2$, $y_1(t_i) = \sqrt{x(t_i)}$ $(i = 1,\ldots,n)$, and $\|y_1\| = \|\sqrt{x}\| = 1$. Setting $y = y_1^2$, we see that $y \in A$, $y \geqslant 0$, $y(t_i) = x(t_i)$ $(i = 1,\ldots,n)$, $\|y\| = 1$, and

$$\|x - y\| = \|x - y_1^2\| \leqslant (\|\sqrt{x}\| + \|y_1\|)\|\sqrt{x} - y_1\| < \epsilon,$$

which completes the proof.

Remark 4.2. It is worth noting that Corollary 4.1 is actually *equivalent* to Theorem 4.1. For we have just shown that Theorem 4.1 implies Corollary 4.1. On the other hand, if Corollary 4.1 is assumed, and $x \in C(T)$, $\epsilon > 0$, write $x = x_1 - x_2$, where $x_1 = \max\{x,0\}$, and $x_2 = \max\{-x,0\}$. By Corollary 4.1, we can choose nonnegative functions y_1 and y_2 in A such that $\|x_i - y_i\| < \epsilon/2$, $y_i(t_j) = x_i(t_j)$ $(j = 1,\ldots,n)$, and $\|y_i\| = \|x_i\|$ $(i = 1,2)$. Setting $y = y_1 - y_2$, we see that $y \in A$, $\|x - y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1,\ldots,n)$, and

$$\|y\| \leqslant \max\{\|y_1\|, \|y_2\|\} = \max\{\|x_1\|, \|x_2\|\} = \|x\|.$$

Thus $(C(T), A, \{\delta_{t_1} \ldots, \delta_{t_n}\})$ has property SAIN.

Remark 4.3. We have already observed (cf. Remark 4.1) that Theorem 4.1 is false, in general, if "dense subalgebra" is replaced by "dense subspace" or even "dense subspace containing constants." We now show that Theorem 4.1

is also false, in general, if the point evaluation functionals are replaced by other functionals. Let $A = \text{span}\,\{x_1, x_2, \ldots\}$, where $x_i(t) \equiv t^i$ $(i = 1, 2, \ldots)$. By the Stone–Weierstrass theorem, A is a dense subalgebra of $C([1,2])$. Let $x^*(x) = \int_1^2 x(t)\,dt$, for all $x \in C([1,2])$. Consider the constant function $x_0(t) \equiv 1 \notin A$. Each $y \in A$ which has the property that $x^*(y) = x^*(x_0) = 1$ must clearly satisfy $\|y\| > 1 = \|x_0\|$. Hence $(C([1,2]), A, \{x^*\})$ does *not* have property SAIN.

We note that in the above example the functional x^* does not attain its norm on $S(A)$. We now give an example where x^* does attain its norm on $S(A)$, yet $(C(T), A, \{x^*\})$ still does not have property SAIN. Let $\mathscr{P}$ denote the set of algebraic polynomials, and define x^* on $C([0,1])$, by $x^*(x) \equiv 2\int_0^{1/2} x(t)\,dt$. Let $x_0 \in C([0,1])$ be such that $x_0(t) = 1$ if $0 \leqslant t \leqslant \frac{1}{2}$, $x_0(1) = 0$, and $\|x_0\| = 1$. Now, x^* attains its norm on $S(\mathscr{P})$ at the constant 1 function (and at no other point of $S(\mathscr{P})$). If $y \in \mathscr{P}$, and $x^*(y) = x^*(x_0) = 1$, then $y(t) = 1$, so that $\|x_0 - y\| \geqslant 1$. Thus $(C([0,1]), \mathscr{P}, \{x^*\})$ does *not* have property SAIN.

Remark 4.4. In view of the above examples, one might be led to conjecture that a necessary condition that $(C(T), A, \{x_1^*, \ldots, x_n^*\})$ have property SAIN, is that $x_i^* \in \text{span}\,\{\delta_t : t \in T\}$ $(i = 1, \ldots, n)$. However, the following is a simple counterexample. Let $\mathscr{P}$ be as in Remark 4.3, and define x^* on $C([0,1])$ by $x^*(x) \equiv \int_0^1 x(t)\,dt$. Clearly, x^* attains its norm at the unique point $1 \in S(\mathscr{P})$. By Theorem 2.1, it follows that $(C([0,1]), \mathscr{P}, \{x^*\})$ has property SAIN.

The following corollary is an immediate consequence of Theorem 4.1 and the Stone–Weierstrass theorem, and it represents a strengthening of the latter.

COROLLARY 4.2. *Let A be a subalgebra of $C(T)$ which separates the points of T, and such that, for each $t \in T$, there is an element of A which does not vanish at t. Then for each $x \in C(T)$, each finite set of points $t_1, \ldots, t_n \in T$, and each $\epsilon > 0$, there exists a $y \in A$ such that $\|x - y\| < \epsilon$, $y(t_i) = x(t_i)$ $(i = 1, \ldots, n)$, and $\|y\| = \|x\|$.*

Proof. By the hypothesis on A, it follows from the Stone–Weierstrass theorem, that A is dense in $C(T)$. An application of Theorem 4.1 now completes the proof.

Recall that a *linear sublattice* of $C(T)$ is a linear subspace L of $C(T)$ with the property that $x \vee y \in L$ and $x \wedge y \in L$ whenever $x, y \in L$, where

$$(x \vee y)(t) = \max\,\{x(t), y(t)\}, \qquad \text{and} \qquad (x \wedge y)(t) = \min\,\{x(t), y(t)\}.$$

THEOREM 4.2. *Let L be a dense linear sublattice of $C(T)$ which contains constants, and let $t_1, \ldots, t_n \in T$. Then $(C(T), L, \{\delta_{t_1}, \ldots, \delta_{t_n}\})$ has property SAIN.*

Proof. Let $x \in C(T)$, and $\epsilon > 0$. We can assume $\|x\| = 1$. By Yamabe's theorem, there exists $y_1 \in L$ such that $\|x - y_1\| < \epsilon$, and $y_1(t_i) = x(t_i)$ $(i = 1, \ldots, n)$. Let e denote the constant 1 function, and set $y = (y_1 \wedge e) \vee (-e)$. Then $y \in L$,

$y(t_i) = x(t_i)$ $(i = 1,\ldots,n)$, and $\|y\| \leqslant 1$. Now, if $|y_1(t)| \leqslant 1$, then $y(t) = y_1(t)$, and so

$$|y(t) - x(t)| = |y_1(t) - x(t)| < \epsilon.$$

If $y_1(t) > 1$, then $y(t) = 1$, and since $x(t) \leqslant 1$,

$$|y(t) - x(t)| \leqslant |y_1(t) - x(t)| < \epsilon.$$

A similar argument shows that $|y(t) - x(t)| < \epsilon$ when $y_1(t) < -1$. Thus, $\|x - y\| < \epsilon$, and the proof is complete.

Remark 4.5. The condition that L contains constants, in Theorem 4.2, cannot be dropped. To see this, let

$$L = \{x \in C([0,1]) : x'(0) \text{ exists, } x'(0) = x(0)\}.$$

It is easy to see that L is a dense linear subspace of $C([0,1])$ which does not contain the constant function e, where $e(t) \equiv 1$. If $y \in L$, and $y(0) = e(0) = 1$, then $y'(0) = y(0) = 1$, and so, $y(t) > 1$ for some $t > 0$; hence $\|y\| > 1 = \|e\|$. Thus, $(C([0,1]), L, \{\delta_0\})$ does *not* have property SAIN. To complete the counterexample, we shall show that L is a sublattice of $C([0,1])$. It suffices to show that if $x,y \in L$, then $x \vee y \in L$. Let $x,y \in L$, and let $s = x \vee y$. We can assume that $x(0) \geqslant y(0)$.

Case 1. $x(0) > y(0)$.

Then $x(t) > y(t)$ in some interval $[0,\delta)$. Thus, $s(t) = x(t)$ for all $t \in [0,\delta)$, and hence $s'(0)$ exists, $s'(0) = x'(0) = x(0) = s(0)$, i.e., $s \in L$.

Case 2. $x(0) = y(0)$ (and hence $x'(0) = y'(0)$).

Given $\epsilon > 0$, choose $\delta > 0$ such that

$$\left|\frac{x(t) - x(0)}{t} - x'(0)\right| < \frac{\epsilon}{2}, \qquad \text{and} \qquad \left|\frac{y(t) - y(0)}{t} - y'(0)\right| < \frac{\epsilon}{2},$$

whenever $0 < t < \delta$. Then, since $s = \frac{1}{2}[x + y + |x - y|]$, we have, for each $t \in (0,\delta)$,

$$\begin{aligned}
&\left|\frac{s(t) - s(0)}{t} - x'(0)\right| \\
&= \left|\frac{1}{2t}[x(t) - x(0) + y(t) - y(0) + |x(t) - y(t)|] - x'(0)\right| \leqslant \\
&\leqslant \tfrac{1}{2}\left|\frac{x(t) - x(0)}{t} - x'(0)\right| + \tfrac{1}{2}\left|\frac{y(t) - y(0)}{t} - y'(0)\right| \\
&\quad + \tfrac{1}{2}\left|\frac{x(t) - y(t)}{t}\right| \\
&< \tfrac{1}{4}\epsilon + \tfrac{1}{4}\epsilon + \tfrac{1}{2}\left|\frac{x(t) - y(t)}{t}\right| = \tfrac{1}{2}\epsilon + \tfrac{1}{2}\left|\frac{x(t) - y(t)}{t}\right|.
\end{aligned}$$

But, for each $t \in (0, \delta)$,

$$\left|\frac{x(t)-y(t)}{t}\right| = \left|\frac{x(t)-x(0)}{t} - x'(0) - \left[\frac{y(t)-y(0)}{t} - y'(0)\right]\right|$$
$$\leqslant \left|\frac{x(t)-x(0)}{t} - x'(0)\right| + \left|\frac{y(t)-y(0)}{t} - y'(0)\right|$$
$$< \epsilon.$$

Hence, it follows that for each $t \in (0, \delta)$,

$$\left|\frac{s(t)-s(0)}{t} - x'(0)\right| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,$$

i.e., $s'(0)$ exists, and $s'(0) = x'(0) = x(0) = s(0)$; hence $s \in L$.

5. An Application in L_p $(1 < p < \infty)$

Let (T, Σ, μ) be a measure space, and for $1 \leqslant p \leqslant \infty$, let L_p denote the Banach space $L_p(T, \Sigma, \mu)$ with the norm $\|x\| = \|x\|_p$ ([2]; p. 121). If $1 < p < \infty$, then $L_p^* = L_q$, where $q^{-1} + p^{-1} = 1$. If $p = 1$, we shall *assume* that (T, Σ, μ) is such that $L_1^* = L_\infty$ (e.g., this will be the case if (T, Σ, μ) is σ-finite). If $1 \leqslant p < \infty$, and $x^* \in L_p$, then by the *representer* of x^* we mean the function $y \in L_q$ such that

$$x^*(x) = \int_T xy \, d\mu, \qquad \text{for all } x \in L_p,$$

and $\|x^*\| = \|y\|_q$.

In this section we shall only be concerned with the case $1 < p < \infty$.

Let M denote the subset of L_p consisting of those functions which vanish off a set of finite measure. M is a dense subspace of L_p for $1 \leqslant p < \infty$ (cf. [2]; p. 125).

Theorem 5.1. *Let $1 < p < \infty$, let M be as above, and let $x_1^*, \ldots, x_n^* \in L_p^*$. Then the following statements are equivalent.*

(1) *$(L_p, M, \{x_1^*, \ldots, x_n^*\})$ has property* SAIN.

(2) *Each x_i^* $(i = 1, \ldots, n)$ attains its norm on $S(M)$.*

(3) *The representer of each x_i^* $(i = 1, \ldots, n)$ vanishes off a set of finite measure.*

Proof. We can assume that $x_i^* \neq 0$ for $i = 1, \ldots, n$.

(1) $\Rightarrow$ (2) is a consequence of Corollary 2.2, since L_p is reflexive.

(2) $\Rightarrow$ (3): Let $y_i \in L_q$ denote the representer of x_i^* $(i = 1, \ldots, n)$, and let $m_i \in S(M)$ denote the point where x_i^* attains its norm, i.e.,

$$\int_T m_i y_i \, d\mu = \|y_i\|_q \qquad (i = 1, \ldots, n).$$

By the condition for equality in Hölder's inequality, it follows that

$$y_i = |m_i|^{p-1} \operatorname{sgn} m_i \quad \text{a.e.} \qquad (i = 1, \ldots, n).$$

In particular, $y_i = 0$ a.e. off a set of finite measure ($i = 1, \ldots, n$). This proves (3).

(3) $\Rightarrow$ (1): Let $y_i \in L_q$ denote the representer of x_i^* ($i = 1, \ldots, n$), and suppose that y_i vanishes off a set T_i of finite measure. Let $x \in L_p$, and $\epsilon > 0$. Let $T_\epsilon \subset \Sigma$ have the property that $\mu(T_\epsilon) < \infty$, and $\int_{T \sim T_\epsilon} |x|^p \, d\mu < \epsilon^p$. Set $T_0 = \left(\bigcup_1^n T_i\right) \cup T_\epsilon$. Then $\mu(T_0) < \infty$, and each y_i vanishes off T_0. Define a function y, by setting $y = x$ on T_0, and $y = 0$ elsewhere. Then $y \in M$,

$$\|x - y\|^p = \int_T |x - y|^p \, d\mu = \int_{T \sim T_0} |x|^p \, d\mu \leqslant \int_{T \sim T_\epsilon} |x|^p \, d\mu < \epsilon^p,$$

$$x_i^*(y) = \int_T y y_i \, d\mu = \int_{T_0} y y_i \, d\mu = \int_{T_0} x y_i \, d\mu = \int_T x y_i \, d\mu$$
$$= x_i^*(x) \qquad (i = 1, \ldots, n),$$

and

$$\|y\|^p = \int_T |y|^p \, d\mu = \int_{T_0} |x|^p \, d\mu \leqslant \int_T |x|^p \, d\mu = \|x\|^p.$$

Thus, $(L_p, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN, and this completes the proof.

COROLLARY 5.1. *Let M be the subspace of l_p $(1 < p < \infty)$ consisting of those elements $x = (\xi_1, \xi_2, \ldots)$ having only finitely many nonzero components ξ_i. Let $x_1^*, \ldots, x_n^* \in l_p^* = l_q$. The following statements are equivalent.*

(1) *$(l_p, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN.*
(2) *Each x_i^* attains its norm on $S(M)$.*
(3) *The representer of each x_i^* has only finitely many nonzero components.*

6. SOME APPLICATIONS IN L_1

We continue to use the terminology and notation introduced in the last section. As stated there, we assume $L_1^* = L_\infty$.

THEOREM 6.1. *Let M denote the dense subspace of L_1 consisting of those functions which vanish off a set of finite measure, and let $x^* \in L_1^*$. Then $(L_1, M, \{x^*\})$ has property SAIN.*

Proof. We can assume $\|x^*\| = 1$. Let $x \in L_1$, and $\epsilon > 0$. We shall show that there is a $y \in M$ such that $\|x - y\| < \epsilon$, $x^*(y) = x^*(x)$, and $\|y\| \leqslant \|x\|$. We can assume that $\|x\| = 1$. If $|x^*(x)| < 1$, the result is a consequence of Lemma 2.4.

Thus, we may suppose that $x^*(x)=1$ (since the case $x^*(x)=-1$ is similar). Letting $y_1 \in L_\infty$ denote the representer of x^*, we have, in particular, that

$$\int_T xy_1 \, d\mu = 1.$$

It follows that $y_1(t) = \operatorname{sgn} x(t)$ (a.e.) where $x(t) \neq 0$. Choose a set T_1 such that $0 < \mu(T_1) < \infty$, and $x \neq 0$ on T_1. In particular, $y_1 \neq 0$ on T_1. Now choose a set $T_2 \supset T_1$ so that $\mu(T_2) < \infty$, and $\int_{T \sim T_2} |x| \, d\mu < \epsilon/2$. Define y as follows:

$$y(t) = \begin{cases} 0 & \text{if } t \notin T_2, \\ x(t) & \text{if } t \in T_2 \sim T_1, \\ x(t) + \delta(t) & \text{if } t \in T_1, \end{cases}$$

where

$$\delta(t) \equiv \frac{1}{\mu(T_1)\, y_1(t)} \int_{T \sim T_2} xy_1 \, d\mu.$$

Note that

$$|\delta(t)| \leqslant \frac{1}{\mu(T_1)} \int_{T \sim T_2} |x| \, d\mu \qquad (t \in T_1).$$

Then,

$$\begin{aligned} x^*(y) &= \int_T yy_1 \, d\mu = \int_{T_1} (x + \delta) y_1 \, d\mu + \int_{T_2 \sim T_1} xy_1 \, d\mu \\ &= \int_{T_2} xy_1 \, d\mu + \int_{T_1} \delta y_1 \, d\mu = \int_{T_2} xy_1 \, d\mu + \int_{T \sim T_2} xy_1 \, d\mu \\ &= \int_T xy_1 \, d\mu = x^*(x), \end{aligned}$$

$$\|x - y\| = \int_{T_1} |\delta| \, d\mu + \int_{T \sim T_2} |x| \, d\mu \leqslant 2 \int_{T \sim T_2} |x| \, d\mu < \epsilon,$$

and

$$\begin{aligned} \|y\| &= \int_T |y| \, d\mu = \int_{T_1} |x + \delta| \, d\mu + \int_{T_2 \sim T_1} |x| \, d\mu \\ &\leqslant \int_{T_2} |x| \, d\mu + \int_{T_1} |\delta| \, d\mu \\ &\leqslant \int_{T_2} |x| \, d\mu + \int_{T \sim T_2} |x| \, d\mu = \int_T |x| \, d\mu = \|x\|. \end{aligned}$$

This completes the proof.

COROLLARY 6.1. *Let M be the subspace of l_1 consisting of those elements $x = (\xi_1, \xi_2, \ldots)$ having only finitely many nonzero components, and let $x^* \in l_1^*$. Then $(l_1, M, \{x^*\})$ has property SAIN.*

In contrast to Corollary 6.1, we now give an example of a dense subspace M of l_1, and an $x^* \in l_1^*$, such that $(l_1, M, \{x^*\})$ does *not* have property SAIN. In fact, we shall verify

PROPOSITION 6.1. *Let* $M = \{y = (\eta_1, \eta_2, \ldots) \in l_1 : \sum_1^\infty n\eta_n = 0\}$, *let* $x = (1,0,0,\ldots)$, *and let* $x^* = (1,-1,1,1,\ldots) \in l_1^* = l_\infty$. *If* $0 < \epsilon \leqslant \frac{1}{4}$, *then there does not exist any* $y \in M$ *such that* $\|x - y\| < \epsilon$, $x^*(y) = x^*(x)$, and $\|y\| = \|x\|$.

Proof. We have already observed that M is a dense subspace in l_1 (cf. Proposition 2.2). If the result is false, then there exists a $y = (\eta_1, \eta_2, \ldots) \in M$ such that $\|x - y\| < \frac{1}{4}$, $x^*(y) = x^*(x) = 1$, and $\|y\| = 1$. It follows that $\eta_1 = 1 - \delta$, for some δ with $0 < \delta < \frac{1}{4}$. Now $\|y\| = 1$ implies that $1 - \delta + \sum_2^\infty |\eta_n| = 1$, i.e.

$$\delta - |\eta_2| = \sum_3^\infty |\eta_n|. \tag{6.1}$$

From the condition $x^*(y) = 1$, we deduce that $1 - \delta - \eta_2 + \sum_3^\infty \eta_n = 1$, i.e.,

$$\delta + \eta_2 = \sum_3^\infty \eta_n. \tag{6.2}$$

Using eqs. (6.1) and (6.2), we get

$$\delta - |\eta_2| \leqslant |\delta + \eta_2| = \left|\sum_3^\infty \eta_n\right| \leqslant \sum_3^\infty |\eta_n| = \delta - |\eta_2|.$$

Thus, equality must hold in this string of inequalities, and, in particular,

$$\delta - |\eta_2| = |\delta + \eta_2| \geqslant 0. \tag{6.3}$$

From eqs. (6.1), (6.2), and (6.3), we get that $\sum_3^\infty |\eta_n| = \sum_3^\infty \eta_n$, from which it follows that $\eta_n \geqslant 0$, for all $n \geqslant 3$. Since $y \in M$, we must have $1 - \delta + \sum_2^\infty n\eta_n = 0$, or

$$1 - \delta + 2\eta_2 + \sum_3^\infty n\eta_n = 0. \tag{6.4}$$

But, by eq. (6.3), we see that $|\eta_2| \leqslant \delta < \frac{1}{4}$, and since $\eta_n \geqslant 0$ for all $n \geqslant 3$,

$$1 - \delta + 2\eta_2 + \sum_3^\infty n\eta_n \geqslant 1 - \delta + 2\eta_2 > 0,$$

which contradicts eq. (6.4). This contradiction completes the proof.

Remark 6.1. We recall that the same subspace M of l_1 which was used in Proposition 6.1, was also used in Proposition 2.1 to obtain a counterexample of a somewhat different flavor.

Remark 6.2. We do not know whether Theorem 6.1 is valid for more than one functional x^*. However, we can prove the following result (Theorem 6.2) in this direction.

An element $y \in L_\infty$ is said to be *eventually constant*, provided that there exists a set T_y such that $\mu(T_y) < \infty$, and $y(t) = \text{const.}$ for $t \notin T_y$.

In the following theorem we shall assume, in addition, that (T, Σ, μ) is σ-finite.

THEOREM 6.2. *Let M be as in Theorem 6.1, and let $x_1^*, \ldots, x_n^* \in L_1^*$. If for each $i = 1, \ldots, n$, the representer for x_1^* is eventually constant, then $(L_1, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN.*

Proof. If $\mu(T) < \infty$, then $M = L_1$, and the result is trivially true. Thus, we can assume $\mu(T) = \infty$. Let $y_i \in L_\infty$ denote the representer for x_i^* $(i = 1, \ldots, n)$, and let T_i be a set such that $y_i(t) = a_i$ for all $t \notin T_i$ $(i = 1, \ldots, n)$. Let $x \in L_1$ and $\epsilon > 0$. We shall construct a $y \in M$ such that $\|x - y\| < \epsilon$, $x_i^*(y) = x_i^*(x)$ $(i = 1, \ldots, n)$, and $\|y\| \leqslant \|x\|$. Using the σ-finiteness of (T, Σ, μ), we can choose a set T_0 so that

$$T_0 \supset \bigcup_1^n T_i, \qquad \mu\left(T_0 \sim \bigcup_1^n T_i\right) > 0,$$

$$\mu(T_0) < \infty, \qquad \text{and} \qquad \int_{T \sim T_0} |x| \, d\mu < \epsilon/2.$$

(Actually, the σ-finiteness was only used to assert

$$\mu\left(T_0 \sim \bigcup_1^n T_i\right) > 0.)$$

Define y as follows:

$$y(t) = \begin{cases} 0 & \text{if } t \notin T_0, \\ x(t) & \text{if } t \in \bigcup_1^n T_i, \\ x(t) + \delta & \text{if } t \in T_0 \sim \bigcup_1^n T_i, \end{cases}$$

where

$$\delta = \frac{1}{\mu\left(T_0 \sim \bigcup_1^n T_i\right)} \int_{T \sim T_0} x \, d\mu.$$

Then $y \in M$,

$$\|x - y\| = \int_{T_0 \sim \bigcup_1^n T_i} |\delta| \, d\mu + \int_{T \sim T_0} |x| \, d\mu \leqslant \int_{T \sim T_0} |x| \, d\mu + \int_{T \sim T_0} |x| \, d\mu < \epsilon,$$

$$\|y\| = \int_{\bigcup_1^n T_i} |x| \, d\mu + \int_{T_0 \sim \bigcup_1^n T_i} |x + \delta| \, d\mu \leqslant \int_{T_0} |x| \, d\mu + \int_{T_0 \sim \bigcup_1^n T_i} |\delta| \, d\mu$$

$$\leqslant \int_{T_0} |x| \, d\mu + \int_{T \sim T_0} |x| \, d\mu = \|x\|,$$

and, for each $i = 1,\ldots,n$,

$$x_i^*(y) = \int_{T_0} yy_i \, d\mu = \int_{\overset{n}{\underset{1}{\cup}} T_k} xy_i \, d\mu + \int_{T_0 \sim \overset{n}{\underset{1}{\cup}} T_k} (x + \delta) y_i \, d\mu$$

$$= \int_{T_0} xy_i \, d\mu + \delta \int_{T_0 \sim \overset{n}{\underset{1}{\cup}} T_k} a_i \, d\mu$$

$$= \int_{T_0} xy_i \, d\mu + a_i \int_{T \sim T_0} x \, d\mu = \int_{T_0} xy_i \, d\mu + \int_{T \sim T_0} xy_i \, du$$

$$= \int_T xy_i \, d\mu = x_i^*(x).$$

This completes the proof.

An element $(\xi_1, \xi_2, \ldots) \in l_\infty$ is eventually constant if there is an index N such that $\xi_N = \xi_{N+1} = \ldots$.

COROLLARY 6.2. *Let M be the subspace of l_1 consisting of the elements having only finitely many nonzero components, and let $x_1^*, \ldots, x_n^* \in l_1^* = l_\infty$. If each x_i^* is eventually constant, then $(l_1, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN.*

REFERENCES

1. F. DEUTSCH, Simultaneous interpolation and approximation in linear topological spaces. *J. SIAM Appl. Math.* **14** (1966), 1180–1190.
2. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators. Part I: General Theory." Interscience, New York, 1958.
3. R. C. JAMES, Characterizations of reflexivity. *Studia Math.* **123** (1964), 199–216.
4. S. PASZKOWSKI, On approximation with nodes. *Rozprawy Mathematyczne* XIV, Warszawa (1957).
5. I. SINGER, Remarque sur un théorème d'approximation de H. Yamabe. *Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.* **26** (1959), 33–34.
6. J. L. WALSH, Interpolation and Approximation by Rational Functions in the Complex Domain. Am. Math. Soc. Colloq. Publ., Providence, 1935.
7. W. WOLIBNER, Sur un polynôme d'interpolation. *Colloq. Math.* **2** (1951), 136–137.
8. H. YAMABE, On an extension of the Helly's theorem. *Osaka Math. J.* **2** (1950), 15–17.
9. S. YOUNG, Piecewise monotone polynomial interpolation. *Bull. Am. Math. Soc.* **73** (1967), 642–643.